Growth of periodic quotients of hyperbolic groups

Rémi Coulon

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Abstract

Let G be a non-elementary torsion-free hyperbolic group. We prove that the exponential growth rate of the periodic quotient G/G^n tends to the one of G as n odd approaches infinity. Moreover we provide an estimate at which the convergence is taking place.

Contents

Introduction

A group G is said to have finite exponent n if for every $g \in G$, $g^n = 1$. At the beginning of the 20^{th} century, W. Burnside settled the following problem (now known as the *Bounded Burnside Problem*) [4]. Is a finitely generated group with finite exponent necessary finite? In order to study this question the natural object to look at is the free Burnside group of rank k and exponent n denoted by $\mathbf{B}_k(n)$. It is the quotient of the free group of rank k denoted by $\mathbf{F}_k$ by the (normal) subgroup $\mathbf{F}_k^n$ generated by the n-th power of all elements of $\mathbf{F}_k$. It is the largest group of rank k and exponent n.

For a long time it was only established that $\mathbf{B}_k(n)$ was finite for some small exponents ($n = 2$ [4], $n = 3$ [4, 16], $n = 4$ [21] and $n = 6$ [13]). The finiteness of $\mathbf{B}_2(5)$ is still open. In 1968, P.S. Novikov and S.I. Adian achieved a breakthrough. In a series of three articles [18], they provided the first examples of infinite free Burnside groups. More precisely they proved the following result.

If $k \geqslant 2$ and n is an odd integer larger than 4381, then $\mathbf{B}_k(n)$ is infinite. Their result has been improved in many directions. In particular S.V. Ivanov [14] and I.G. Lysenok [17] solved the case of even exponents. Since free Burnside groups of sufficiently large exponents are infinite a natural question is how "big" they are. This can be measured by the exponential growth rate.

Given a finitely generated group G endowed with the word metric with respect to some finite generating set of G, its *(exponential) growth rate* is defined to be

$$\lambda = \lim_{r \to \infty} \sqrt[r]{|B\,(r)|},$$

where $|B\,(r)|$ denotes the cardinal of the ball of radius r of G. If $\lambda > 1$ one says that G has *exponential growth.* (λ depends on the generating set, however having exponential growth is a property of the group G). Furthermore if for every generating set the corresponding growth rate is uniformly bounded away from 1 then G has *uniform exponential growth.*

In his book [1], S.I. Adian proved that free Burnside groups of sufficiently large odd exponents are not only infinite but also exponentially growing. Latter D. Osin showed that they are uniformly non-amenable, and therefore they have uniform exponential growth [20]. An other approach can be found in [3].

In 1991, using a diagrammatical description of graded small cancellation theory, A.Y. Ol'shanskiĭ proved an analogue for hyperbolic groups of the Novikov-Adian Theorem [19].

Theorem 1 (Ol'shanskiĭ [19])**.** *Let G be a non-elementary, torsion-free hyperbolic group. There exists a critical exponent $n(G)$ such that for all odd integers $n \geqslant n(G)$, the quotient G/G^n is infinite.*

Non-elementary hyperbolic groups are known to have uniform exponential growth [15]. On the other hand hyperbolic groups are growth tight [2]. This means that, given such a group G and a finite generating set A, for any infinite normal subgroup N of G, the exponential growth rate of G/N with respect to the natural image of A is strictly less than the exponential growth rate of G with respect to A. Therefore we were wondering what the growth rate of the periodic quotients G/G^n could be. In particular is there a gap between the respective growth rates of G and G/G^n? The following theorem answers this question negatively: the growth rate of G/G^n converges to the one of G as n odd approaches infinity. Moreover we provide an estimate for the rate at which this convergence is taking place.

Theorem 2. *Let G be a non-elementary, torsion-free hyperbolic group and λ its exponential growth rate with respect to a finite generating set A of G. There exists a positive number κ such that for sufficiently large odd exponents n the exponential growth rate of G/G^n with respect to the image of A is larger than*

$$\lambda \left(1 - \frac{\kappa}{n}\right).$$

In the case of free Burnside groups we even have a much more accurate estimate.

Theorem 3. *Let $k \geqslant 2$. Let A be a free generating set of $\mathbf{F}_k$ (i.e. with exactly k elements). There exists a positive number κ such that for sufficiently large odd exponents n the exponential growth rate of $\mathbf{B}_k(n)$ with respect of the image of A is lager than*

$$(2k-1)\left(1 - \frac{\kappa}{(2k-1)^{n/2}}\right).$$

Our proof extends the ideas of S.I. Adian. However considering hyperbolic groups instead of free groups makes it much more complicated and requires new tools. Let us first recall the key argument of Adian's approach.

Main fact. Let v be a reduced word representing an element of $\mathbf{F}_k$. If v does not contain a subword of the form w^{16}, then v induces a non-trivial element of $\mathbf{B}_k(n)$ for every odd integer $n \geqslant 665$.

In particular, two distinct reduced words which do not contain a 8-th power induce different elements of $\mathbf{B}_k(n)$. Therefore, it is sufficient to estimate the growth rate of F_8, the set of reduced words without 8-th power. This is done by induction on the length of the words. The main steps of this proof are recalled in Section 3.1.

Consider now an arbitrary non-elementary, torsion-free, hyperbolic group G endowed with the word metric $|\ .\ |$. Following A.Y. Ol'shanskiĭ, a (L,m)-power is an element of G that can be written uw^nu' where u and u' have length at most L. An element g of G is (L,m)-aperiodic if it can not be written $g = g_1g_2g_3$ where $|g| = |g_1| + |g_2| + |g_3|$ and g_2 is a (L,m)-power. The proof of Theorem 1 relies on the following fact. In [19], A.Y. Ol'shanskiĭ proved the existence of constants L, ε and $n(G)$, which only depends on G with the following property. Let n be an odd integer larger than $n(G)$, let $m \leqslant \varepsilon n$. Then the set of (L,m)-aperiodic elements embeds into G/G^n. The infiniteness of G/G^n follows from the one of (L,m)-aperiodic elements. An other approach based on techniques developed by T. Delzant and M. Gromov [10] can be found in [8].

Hence one way to prove Theorem 2 is to compute the growth rate of the set of (L,m)-aperiodic elements. Instead of reasoning with words we consider geodesic path the Cayley graph X of G. However this definition of (L,m)-aperiodic words does not behave well with the operations of extending geodesics or taking subgeodesics. For instance we would like to have the following fact. Let g and g' be two element of G such that g lies on a geodesic between 1 and g. If g is (L,m)-aperiodic but g' is not, then the m-th power in g' can be read "at the end" of the geodesic representing g'. Nevertheless, since X is not uniquely geodesic, this statement does not hold: the element g could contain a $(L+4\delta,m)$-power as illustrated on Figure 1.

To avoid this difficulty we focus on a particular set of geodesics. We fix an arbitrary order on the generating set $A \cup A^{-1}$. Thus the set of geodesics inherits of the lexicographical order. For every $g \in G$, σ_g is the smallest (for the lexicographical order) geodesic joining 1 to g. We call such a path a *lexicographic geodesic*. In particular, if $g' \in G$ lies on σ_g then $\sigma_{g'}$ is the subpath of σ_g between

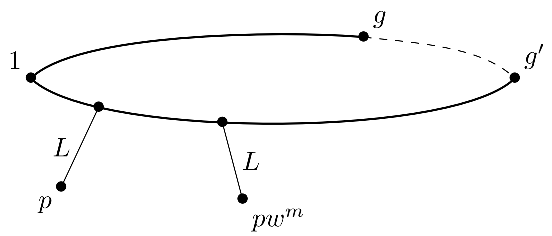

Figure 1: Extending aperiodic elements.

1 and g'. Then we adopt the following definition. An element $g \in G$ contains a (L, m)-power if there are $p \in G$ and a non-trivial cyclically reduced element $w \in G$ such that both p and pw^m belong to the L-neighborhood of σ_g.

This adaptation leads to an other difficulty. Given an element $g \in G$, we need to be sure that σ_g can be extended in "sufficiently many ways" in a lexicographic geodesic. However this could be impossible (see Fig. 2). This question is handled in Section 2. For every r we construct a subset F of G which, among others, satisfies the following property. For all $g \in F$ the number of elements $g' \in F$ such that $\sigma_{g'}$ extends σ_g by a length r is larger that $\nu\lambda^r$, where ν is some constant which only depends on G and A and λ is the exponential growth rate of G. Our proof uses as a tool the Canon cone types [5].

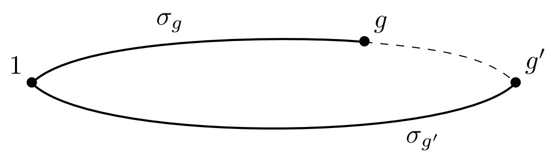

Figure 2: Extending lexicographic geodesics.

Finally we prove that the set of (L, m)-aperiodic elements of F grows exponentially with a rate larger than $\lambda(1-\kappa/n)$ (see Section 3). Our theorem follows then from Ol'shanskiĭ's work (see Section 4).

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1 Hyperbolic geometry

In this section we fix notations and review some of the standard facts on hyperbolic spaces and hyperbolic groups (in the sense of Gromov). For more details we refer the reader to the original paper of M. Gromov [12] or to [7, 11].

Let G be a group generated by a finite set A. We denote by X the Cayley graph of G with respect to A. The vertices of X are the elements of G. For every $g \in G$ and $a \in A \cup A^{-1}$, g is joined to ga by an edge labeled by a. The group G acts on the left by isometries on X.

Given two points $x, x' \in X$, $|x - x'|$ stands for the distance between them. The Gromov product of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2} \left\{ |x - z| + |y - z| - |x - y| \right\}.$$

The space X is said to be δ-hyperbolic if for all $x, y, z, t \in X$,

$$\langle x, y \rangle_t \geqslant \min \left\{ \langle x, z \rangle_t , \langle z, y \rangle_t \right\} - \delta.$$

Remark : The constant δ depends on A. Nevertheless for a group, being hyperbolic (for some δ) does not depend on the generating set. In this article we fix once for all the generating set A. Therefore the hyperbolicity constant δ of X is fixed as well. Without loss of generality we can assume that $\delta \geqslant 1$. More generally, greek letters will represent constants which only depend on G and A. Moreover in the rest of the article we assume that G is torsion-free and non-elementary i.e., non virtually cyclic.

As a consequence of hyperbolicity, the geodesic triangles of X are 4δ-thin i.e., for every $x, y, z \in X$, for every p (respectively q) lying on a geodesic between x and y (respectively between x and z), if $|x - p| = |x - q| \leqslant \langle y, z \rangle_x$ then $|p - q| \leqslant 4\delta$.

Let $g \in G$. For simplicity of notation, $|g|$ stands for the distance $|g - 1|$. This is exactly the word length of g with respect to A. To measure the action of g on X we define two quantities: the *translation length* $[g]$ and the *stable translation length* $[g]^\infty$.

$$[g] = \inf_{x \in X} |gx - x| \, ; \quad [g]^\infty = \lim_{n \to \infty} \frac{1}{n} |g^n x - x|$$

They are related by the following inequality: $[g]^\infty \leqslant [g] \leqslant [g]^\infty + 50\delta$. A *cyclically reduced isometry* is an element $g \in G$ such that $[g] = |g|$. Every conjugacy class of G contains such an isometry. The set of all non-trivial cyclically reduced isometries is denoted by C.

If $[g]^\infty > 0$ then g is called *hyperbolic*. It is known that every element of G is either hyperbolic or has finite order (in the latter case it is said to be *elliptic*). For our purpose we assumed that G was torsion-free. Therefore every non-trivial isometry is hyperbolic.

In a hyperbolic group, the range of stable translation lengths is discrete:

Theorem 1.1 (Delzant, [9])**.** *There exists a constant $\tau \in \mathbf{Q}_+^*$ such that for all $g \in G$, $[g]^\infty \in \tau\mathbf{N}$.*

In particular, for every $g \in G$, hyperbolic we have $[g]^\infty \geqslant \tau$.

Given $r \in \mathbf{R}_+$ and $g \in G$, we denote by $B(g,r)$ the close ball of G of center g and radius r i.e., the set of $h \in G$ such that $|g-h| \leqslant r$. If g is the trivial element we simply write $B(r)$. For all $e \in \mathbf{R}_+$ the *annulus* $A(r,e)$ is the set of elements $g \in G$ such that $r-e \leqslant |g| \leqslant r$. If $r \geqslant 0$ and $e \geqslant 1$, then $A(r,e)$ is not empty.

If P is a finite subset of G, $|P|$ stands for its cardinal. In order to estimate the size of an infinite subset of G we use the exponential growth rate

Definition 1.2. *Let P be a subset of G. The* (exponential) growth rate *of P is the quantity*

$$\limsup_{r\to+\infty} \sqrt[r]{|P \cap B(r)|}$$

We denote by λ the growth rate of G. Since the map $r \to |B(r)|$ is submultiplicative, λ satisfies in fact

$$\lambda = \lim_{r\to+\infty} \sqrt[r]{|B(r)|} = \inf_{r\in\mathbf{N}^*} \sqrt[r]{|B(r)|}.$$

In particular for all $r \in \mathbf{N}$, $|B(r)| \geqslant \lambda^r$. The next proposition gives an upper bound for $|B(r)|$.

Proposition 1.3 (Coornaert, [6])**.** *There exists $\alpha \geqslant 1$ which only depends on G and A such that for all $r \in \mathbf{R}_+$, $|B(r)| \leqslant \alpha\lambda^r$.*

2 Growth of cone types

2.1 Essential cone types

Definition 2.1. *Let $g \in G$. The* cone type *of g is the set of elements $u \in G$ such that there exists a geodesic of X between 1 and gu that passes through g. We denote it by T_g.*

Since G is a hyperbolic group, the set of all cone types, denoted by $\mathcal{T}$, is finite [7, Chap. 12, Th. 3.2].

Definition 2.2. *We say that a cone type $T \in \mathcal{T}$ is* essential *if*

$$\exists a > 0,\ \forall r \geqslant 0,\ \exists s \geqslant r, \quad |T \cap B(s)| \geqslant a\lambda^s,$$

where λ is the growth rate of G.

Notation : An element $g \in G$ is *essential* if its cone type T_g is essential. The set of all essential elements is denoted by E. We write $\mathcal{T}_E$ for the set of all essential cone types.

Remark : It follows easily from the definition that the growth rate of an essential cone type is exactly λ. Roughly speaking, the essential elements are the ones who are responsible for the growth of G.

Proposition 2.3. *There exists $\beta > 0$ (which only depends on G and A) such that for all $T \in \mathcal{T}_E$, for all $r \in \mathbf{R}_+$,*

$$|T \cap B(r)| \geqslant \beta\lambda^r.$$

Proof. Note that $\mathcal{T}_E$ is finite. Hence it is sufficient to prove the following statement:

$$\forall T \in \mathcal{T}_E,\ \exists \beta > 0,\ \forall r \geqslant 0,\ |T \cap B(r)| \geqslant \beta \lambda^r.$$

Let T be an essential type. By definition there exists $a > 0$ such that for all $r \geqslant 0$ there is $s \geqslant r$ such that $|T \cap B(s)| \geqslant a\lambda^s$. Let $r \in \mathbf{R}_+$. We denote by s a real number larger than r such that $|T \cap B(s)| \geqslant a\lambda^s$. Every element of $T \cap B(s)$ can be written uv where $u \in T \cap B(r)$ and $v \in B(s-r+1)$. (Note that r is not necessary an integer, therefore v is a ball of radius $s-r+1$ and not $s-r$.) Consequently $|T \cap B(s)| \leqslant |T \cap B(r)|\,|B(s-r+1)|$. Using Proposition 1.3 we obtain

$$a\lambda^s \leqslant |T \cap B(s)| \leqslant \alpha \lambda^{s-r+1} |T \cap B(r)|.$$

Thus for all $r \geqslant 0$, $|T \cap B(r)| \geqslant \alpha^{-1}\lambda^{-1} a \lambda^r$. □

Lemma 2.4. *Let $g \in G$. Let $u \in T_g$. If gu is essential then so is g.*

Proof. By definition of the cone type, uT_{gu} is a subset of T_g. Hence for all $r \geqslant |u|$, $|T_g \cap B(r)| \geqslant |T_{gu} \cap B(r-|u|)|$. However gu is essential. It follows from Proposition 2.3 that for all $r \geqslant |u|$,

$$|T_g \cap B(r)| \geqslant |T_{gu} \cap B(r-|u|)| \geqslant \beta \lambda^{-|u|} \lambda^r.$$

Thus g is essential. □

Let $g \in G$ and $u \in T_g$. According to the previous lemma, if gu is essential, so is g. The converse statement is not necessary true. Nevertheless, the next proposition gives a lower bound for $\left|T_g \cap g^{-1}E \cap A(r,e)\right|$ which is the number of elements $u \in T_g \cap A(r,e)$ such that gu is essential. (Recall that $A(r,e)$ is the annulus of radius r defined in Section 1.)

Proposition 2.5. *There exists $\gamma > 0$ (which only depends on G and A) such that for all $g \in E$ and for all $r \geqslant 0$, for all $e \geqslant 1$.*

$$\left|T_g \cap g^{-1}E \cap A(r,e)\right| \geqslant \gamma \lambda^r.$$

Proof. Note that it is sufficient to prove the proposition for $e = 1$. Let $\gamma > 0$. Suppose the proposition were false. There exists an essential element $g \in E$ and $r \in \mathbf{R}_+$ such that $\left|T_g \cap g^{-1}E \cap A(r,1)\right| < \gamma\lambda^r$. Negating the definition of essential types, we have

$$\forall T \in \mathcal{T} \setminus \mathcal{T}_E,\ \exists s \geqslant 0,\ \forall t \geqslant s, \quad |T \cap B(t)| < \gamma \lambda^t.$$

Recall that the set of cone types $\mathcal{T}$ is finite. Thus we have in fact

$$\exists s \geqslant 0,\ \forall T \in \mathcal{T} \setminus \mathcal{T}_E,\ \forall t \geqslant s, \quad |T \cap B(t)| < \gamma \lambda^t. \tag{1}$$

Let $t \geqslant s$. It follows from the definition of cone types that

$$T_g \cap B(r+t) \subset T_g \cap B(r-1) \cup \left(\bigcup_{u \in T_g \cap A(r,1)} u\left(T_{gu} \cap B(t+1)\right) \right).$$

Since g is essential, Proposition 2.3 yields

$$\beta\lambda^{r+t} \leqslant |T_g \cap B(r+t)| \leqslant |T_g \cap B(r-1)| + \sum_{u \in T_g \cap A(r,1)} |T_{gu} \cap B(t+1)|.$$

Let $u \in T_g \cap A(r,1)$. If u does not belong to $g^{-1}E$, then gu is not essential. By (1), $|T_{gu} \cap B(t+1)| \leqslant \gamma\lambda^{t+1}$. On the other hand, if $u \in g^{-1}E$, then Proposition 1.3 leads to $|T_{gu} \cap B(t+1)| \leqslant |B(t+1)| \leqslant \alpha\lambda^{t+1}$. It follows that $|T_g \cap B(r+t)|$ is bounded above by

$$|T_g \cap B(r-1)| + \gamma\lambda^{t+1} \left|T_g \cap A(r,1) \setminus g^{-1}E\right| + \alpha\lambda^{t+1} \left|T_g \cap g^{-1}E \cap A(r,1)\right|.$$

However by assumption, $\left|T_g \cap g^{-1}E \cap A(r,1)\right| \leqslant \gamma\lambda^r$. Moreover Proposition 1.3 gives $|T_g \cap B(r-1)| \leqslant \alpha\lambda^{r-1}$ and $\left|T_g \cap A(r,1) \setminus g^{-1}E\right| \leqslant \alpha\lambda^r$. Thus for all $t \geqslant s$, $\beta\lambda^{r+t} \leqslant \alpha\lambda^{r-1} + 2\alpha\gamma\lambda^{r+t+1}$. Therefore $0 < \beta \leqslant 2\alpha\lambda\gamma$. This inequality holds for all $\gamma > 0$, which is impossible. $\square$

2.2 Lexicographic types

Recall that A is a finite generating set of G and X the Cayley graph of G with respect to A. In this section we fix an arbitrary order on $A \cup A^{-1}$. Let $g, h \in G$. Using the labeling of the edges of X, any geodesic joining g to h can be identify with a word over the alphabet $A \cup A^{-1}$ representing hg^{-1}. Thus the set of geodesics inherits from the lexicographic order. (We read the words from the left to the right.) For all $g \in G$, we denote by σ_g the geodesic joining 1 to g which is the smallest for the lexicographic order. We call it the *lexicographic geodesic* from 1 to g. Note that if $h \in G$ lies on σ_g then σ_h is exactly the subpath of σ_g between 1 and h.

Definition 2.6. *Let $g \in G$. The* lexicographic type *of g is the set of elements $u \in G$ such that σ_{gu} passes through g. We denote it by L_g.*

Remark : It follows from the definition that L_g is a subset of T_g. Contrary to $\mathcal{T}$, it is not clear whether or not the set of all lexicographic types is finite.

Our goal is to construct a subset F of G such that its elements satisfy analogues for the lexicographic types of Proposition 2.5 and Lemma 2.4. To that end, we need the following lemma.

Lemma 2.7 (Arzhantseva-Lysenok, [2, Lemma 5])**.** *There exists a constant $\rho > 0$ which only depends on G and A satisfying the following. For every finite subset P of G there is a subset P' of P such that $|P'| \geqslant \rho|P|$ and for all distinct $g, g' \in P'$, $|g - g'| > 20\delta$.*

Recall that γ is the constant given by Proposition 2.5. Let us put $\nu = \rho\gamma\lambda^{-4\delta}|B(4\delta)|^{-1}$. This number only depends on G and A. Let r be a real number larger than 10δ. The set F that we are going to build will depend on the parameter r, which represents a distance. However for simplicity, we do not mention the dependence on r in the notation. First, we construct by induction a non-increasing sequence (H_i) of subsets of G.

- Put $H_0 = G$.

- Let $i \in \mathbf{N}$. Assume that H_i is already defined. The set H_{i+1} is

$$H_{i+1} = \left\{ g \in H_i \,\middle|\, \left| L_g \cap g^{-1} H_i \cap A\left(r, 9\delta\right) \right| \geqslant \nu \lambda^r \right\}$$

The set H is defined to be the intersection of all H_i's.

Lemma 2.8. *Let $g \in G$. Let $i \in \mathbf{N}$. If $\left| L_g \cap g^{-1} H_i \cap A\left(r, 9\delta\right) \right| \geqslant \nu \lambda^r$ then g belongs to H_{i+1}.*

Proof. Let $j \leqslant i$. By construction H_i is a subset of H_j. Therefore

$$\left| L_g \cap g^{-1} H_j \cap A\left(r, 9\delta\right) \right| \geqslant \left| L_g \cap g^{-1} H_i \cap A\left(r, 9\delta\right) \right| \geqslant \nu \lambda^r.$$

A proof by induction on j shows that for all $j \leqslant i+1$, g belongs to H_j. □

Proposition 2.9. *For all $i \in \mathbf{N}$, for all $g \in E$, $B(g, 4\delta) \cap H_i$ is non-empty.*

Proof. We prove this proposition by induction on i. If $i = 0$, the proposition follows from the fact that $H_0 = G$. Assume now that the proposition holds for $i \in \mathbf{N}$. Let $g \in E$. According to Proposition 2.5, $\left| T_g \cap g^{-1} E \cap A\left(r - 4\delta, \delta\right) \right| \geqslant \gamma \lambda^{r-4\delta}$. By Lemma 2.7, there exists a subset P of $T_g \cap g^{-1} E \cap A\left(r - 4\delta, \delta\right)$ such that

(i) $|P| \geqslant \rho \left| T_g \cap g^{-1} E \cap A\left(r - 4\delta, \delta\right) \right| \geqslant \rho \gamma \lambda^{r-4\delta}$,

(ii) for all distinct $u, u' \in P$, $|u - u'| > 20\delta$.

Let $u \in P$. The isometry gu is essential. Applying the induction assumption $B(gu, 4\delta) \cap H_i$ contains an element that we shall write $h_u = lv$ where l belongs to G with $|g| = |l|$ and v to L_l. By construction l (respectively g) belongs to a geodesic between 1 and lv (respectively gu), thus

$$\left| |u| - |v| \right| = \left| |gu| - |lv| \right| \leqslant |gu - lv| \leqslant 4\delta.$$

Since u lies in $A\left(r - 4\delta, \delta\right)$, v is an element of $A\left(r, 9\delta\right)$. Moreover $|gu - g| \geqslant r - 5\delta \geqslant |lv - gu|$. By hyperbolicity $|l - g| \leqslant 4\delta$. Consequently $\{h_u | u \in P\}$ is a subset of $\bigcup_{l \in B(g, 4\delta)} l\left(L_l \cap A\left(r, 9\delta\right)\right)$. On the other hand we claim that $\{h_u | u \in P\}$ is a subset of H_i that contains exactly $|P|$ elements. Let $u, u' \in P$ such that $h_u = h_{u'}$. By triangle inequality we have

$$|u - u'| = |gu - gu'| \leqslant |gu - h_u| + |h_{u'} - gu'| \leqslant 8\delta.$$

By definition of P, $u = u'$. Therefore $\bigcup_{l \in B(g, 4\delta)} l\left(L_l \cap A\left(r, 9\delta\right)\right)$ contains at least $|P|$ elements of H_i. Hence there is $l \in B(g, 4\delta)$ such that

$$\left| L_l \cap l^{-1} H_i \cap A\left(r, 9\delta\right) \right| \geqslant \left| B\left(4\delta\right) \right|^{-1} |P| \geqslant \rho\gamma \left| B\left(4\delta\right) \right|^{-1} \lambda^{r-4\delta} = \nu \lambda^r.$$

It follows from Lemma 2.8, that l belongs to $B(g, 4\delta) \cap H_{i+1}$. Thus the proposition holds for $i + 1$. □

Corollary 2.10. *For all $g \in E$, $B(g, 4\delta) \cap H$ is non-empty.*

Corollary 2.11. *For all $g \in H$, $\left| L_g \cap g^{-1} H \cap A\left(r, 9\delta\right) \right| \geqslant \nu \lambda^r$.*

Proof. Both corollaries follow from the fact that $H = \bigcap_{i \in \mathbf{N}} H_i$. □

Remark : In particular the set H is non empty. Since 1 is essential ($T_1 = G$) Corollary 2.10 tells us that $B(4\delta)$ contains an element of H. Actually, the same proof as the one of Proposition 2.9 shows that 1 belongs to H.

Corollary 2.11 is an analogue for the lexicographic types of Proposition 2.5. However given $g \in G$ and $u \in L_g$ such that gu belongs to H there is no reason that g should also belong to H. That is why we have to consider a subset F of H which will in addition satisfy an analogue of Lemma 2.4 (see Lemma 2.13). To that end we proceed by induction

- Put $F_0 = \{1\}$.
- Let $i \in \mathbf{N}$ such that F_i is already defined. The set F_{i+1} is given by

$$F_{i+1} = \bigcup_{g \in F_i} g\left(L_g \cap g^{-1}H \cap A(r, 9\delta)\right).$$

Finally, the set F is the union of all the F_i's. Note that F is a subset of H. Moreover for all $g \in F$, $L_g \cap g^{-1}H \cap A(r, 9\delta)$ lies inside $g^{-1}F$. Therefore Corollary 2.11 leads to the following result.

Lemma 2.12. *For all $g \in F$, $\left|L_g \cap g^{-1}F \cap A(r, 9\delta)\right| \geqslant \nu\lambda^r$.*

Lemma 2.13. *Let $g \in F$ and x be a point of σ_g. There exists $h \in F$ which lies on σ_g between 1 and x such that $|x - h| \leqslant r$.*

Proof. Let us denote by i, the smallest integer such that x is on a geodesic σ_l with $l \in F_i \cap \sigma_g$. If $i = 0$ then $l = x = 1$. Thus the lemma is obvious. Assume now that $i \geqslant 1$. By definition there exists $h \in F_{i-1}$ such that l belongs to $h\left(L_h \cap h^{-1}H \cap A(r, 9\delta)\right)$. In particular h and l are two points of σ_g. We claim that x lies on σ_g between h and l. Suppose it were false, then x would lie on σ_h which contradicts the minimality of i. However, by construction $|l - h| \leqslant r$ Consequently h is a point of $F \cap \sigma_g$ between 1 and x such that $|x - h| \leqslant |l - h| \leqslant r$. □

Finally we have proved the following result.

Proposition 2.14. *There is $\nu > 0$ (which only depends on G and A) such that for all $r \geqslant 10\delta$, there exists a subset F of G satisfying the following properties.*

(i) 1 belongs to F,

(ii) for all $g \in F$, $\left|L_g \cap g^{-1}F \cap A(r, 9\delta)\right| \geqslant \nu\lambda^r$,

(iii) for all $g \in F$, for all $x \in \sigma_g$, there exists $h \in F$ which lies on σ_g between 1 and x such that $|x - h| \leqslant r$.

3 Avoiding large powers

The goal here is to estimate the growth rate of a subset of G “without” power. This section involves many parameters. As a warmup we start with the case of free groups. We present briefly the idea used by S.I. Adian in [1]. The estimation that we obtain in that particular case will also be useful in Section 4.

3.1 The case of free groups

In this section we assume that A is a free generating set of $\mathbf{F}_k$ i.e., it contains exactly k elements. Consequently the exponential growth rate of $\mathbf{F}_k$ with respect to A is $\lambda = 2k-1$. Let $m \in \mathbf{N}$. An element of $\mathbf{F}_k$ is said to be m-aperiodic, if the reduced word over the alphabet $A \cup A^{-1}$ representing it does not contain a subword of the form u^m. We denote by F_m the set of m-aperiodic elements of $\mathbf{F}_k$

Proposition 3.1. *For all integers $m \geqslant 2$, for all $s \in \mathbf{N}$,*

$$|F_m \cap B(s+1)| \geqslant \lambda |F_m \cap B(s)| - \frac{2k}{2k-1} \sum_{j \geqslant 1} \lambda^j |F_m \cap B(s+1-mj)|.$$

Proof. Let $s \in \mathbf{N}$. An m-aperiodic word w of length $s+1$ can be written $w = w'a$ where w' is an m-aperiodic word of length s and a an element of $A \cup A^{-1}$. Since w is reduced the number of possible choices for a is $2k-1$. On the other hand, consider a reduced of the form $w'a$ where $w' \in F_m \cap B(s)$ and $a \in A \cup A^{-1}$. If such a word is not m aperiodic, then there exists $j \in \mathbf{N}^*$, $u \in \mathbf{F}_k$ with $|u| = j$ and $p \in F_m \cap B(s+1-mj)$ such that $w'a = pu^m$. The number of words of this last form is bounded above by

$$|F_m \cap B(s+1-mj)| \,.\, |A(j,0)| \leqslant 2k(2k-1)^{j-1} |F_m \cap B(s+1-mj)|.$$

Therefore the number of reduced words of the form $w'a$ ($w' \in F_m \cap B(s)$ and $a \in A \cup A^{-1}$) which are m-aperiodic is bounded below by

$$(2k-1) |F_m \cap B(s)| - 2k \sum_{j \geqslant 1} (2k-1)^{j-1} |F_m \cap B(s+1-mj)|,$$

which gives the desired conclusion. □

Proposition 3.2. *Let $k \geqslant 2$. For every $a > 2k$, there exists an number m_0 such that for every integer $m \geqslant m_0$ the exponential growth rate of F_m is larger than $\lambda(1 - a\lambda^{-m})$.*

Proof. We consider the function $f_m : \left(\sqrt[m]{\lambda}, \lambda\right) \to \mathbf{R}$ defined by

$$f_m(\mu) = \lambda - \frac{2k\mu}{2k-1} \sum_{j \geqslant 1} \left(\frac{\lambda}{\mu^m}\right)^j = \lambda - \frac{2k\mu}{\mu^m - \lambda}$$

Let $a > 2k$. We put $\mu_m = \lambda(1 - a\lambda^{-m})$. The sequence μ_m tends to $\lambda > 1$ as m approaches infinity. Therefore

$$f_m(\mu_m) = \lambda \left(1 - \frac{2k}{\lambda^m}\right) + \underset{m \to +\infty}{\mathrm{o}}(1).$$

Since $a > 2k$ there exists a number m_0 such that for every integer $m \geqslant m_0$, $f_m(\mu_m) \geqslant \mu_m$. Fix $m \geqslant m_0$. For simplicity of notation we write μ for μ_m. We now prove by induction that for every $s \in \mathbf{N}$, $|F_m \cap B(s)| \geqslant \mu |F_m \cap B(s-1)|$. The statement is true for $s = 0$. Assume that it holds for all integers less or equal

to s. In particular for every $j \geqslant 1$, $|F_m \cap B(s+1-mj)| \leqslant \mu^{1-mj} |F_m \cap B(s)|$. It follows then from Proposition 3.1 that

$$|F_m \cap B(s+1)| \geqslant \left[\lambda - \frac{2k\mu}{(2k-1)} \sum_{j \geqslant 1} \left(\frac{\lambda}{\mu^m} \right)^j \right] |F_m \cap B(s)| \, .$$

The expression between the brackets is exactly $f_m(\mu)$. Therefore the assumption holds for $s+1$. A second induction proves that for every $s \in \mathbf{N}$, $|F_m \cap B(s)| \geqslant \mu^s$, which leads to the result. $\square$

3.2 The general case

We now deal with the case of hyperbolic groups. Recall that a cyclically reduced isometry is an element $g \in G$ such that $[g] = |g|$. The set of non-trivial cyclically reduced elements is denoted by C (see Section 1). Let $L > 0$ and $m \in \mathbf{N}^*$. Given $g \in G$, we say that g contains a (L, m)-*power*, if there exists $(l, w) \in G \times C$ such that both l and lw^m belong to the L-neighbourhood of σ_g. If g does not contain any (L, m)-power it is called (L, m)-*aperiodic*.

Remark : Our definition of (L, m)-aperiodic elements is a slightly weaker form of the one of A.Y. Ol'shanskiĭ [19]. However it is sufficient to apply Ol'shanskiĭ's results (see the remark following Theorem 4.4).

Let $g \in G$ and h be an element of G which lies on σ_g. By construction, the geodesic σ_h is the subpath of σ_g joining 1 and h. Therefore if g is (L, m)-aperiodic, so is h.

Given a subset F of G, we denote by $F_{L,m}$ the set of elements $g \in F$ which are (L, m)-aperiodic. Our aim is to give a lower bound for the growth rate of $F_{L,m}$ for an appropriate subset F of G. More precisely we prove the following result.

Proposition 3.3. *Let $L > 0$. There exist $a > 0$ and $m_0 \in \mathbf{N}$ satisfying the following property. For all integers $m \geqslant m_0$, there is a subset F of G, such that the exponential growth rate of $F_{L,m}$ is larger than $\lambda(1 - a/m)$.*

The rest of this section is dedicated to the proof of Proposition 3.3. We first need to fix some parameters. The constants τ and ν are the ones respectively given by Theorem 1.1 and Proposition 2.14. First we take $L > 0$. Its value will be made precise in the next section (see Theorem 4.4). Let r be a real number larger than 10δ and m an integer such that $m\tau \geqslant 2L + r$. According to Proposition 2.14, there exists a subset F of G containing 1, such that

(i) for all $g \in F$, $\left| L_g \cap g^{-1} F \cap A(r, 9\delta) \right| \geqslant \nu \lambda^r$,

(ii) for all $g \in F$, for all $x \in \sigma_g$, there exists $h \in F$ which lies on σ_g between 1 and x such that $|x - h| \leqslant r$.

We now define auxiliary subsets of G.

- $Z = \left\{ hu \;\middle|\; h \in F_{L,m}, u \in L_h \cap h^{-1} F \cap A(r, 9\delta) \right\}$. (Note that $Z \subset F$.)

- For all $w \in G \setminus \{1\}$, Z_w is the set of elements $g \in G$ that can be written $g = huw^m u'$ where
  - (i) h belongs to $F_{L,m}$,
  - (ii) $|g| \geqslant |h| + |w^m| - 2L$.
  - (iii) $|u|, |u'| \leqslant L + r$.

Roughly speaking Z denotes the set of elements of F which geodesically extend a (L,m)-aperiodic isometry by a length r. On the other hand, Z_w contains the elements which extend a (L,m)-aperiodic isometry by a m-th power of w. The idea of the next proposition is the following. An (L,m)-aperiodic isometry of length $s+r$ can be obtained by extending a (L,m)-aperiodic isometry of length s. However this extension should not involve a m-th power.

Proposition 3.4. *The set $Z \setminus \bigcup_{w\in C} Z_w$ is contained in $F_{L,m}$.*

Proof. Equivalently we prove that $Z \setminus F_{L,m} \subset \bigcup_{w\in C} Z_w$. Let g be an element of $Z \setminus F_{L,m}$. In particular it belongs to $F \setminus F_{L,m}$. Therefore there exist $l \in G$ and $w \in C$ such that l and lw^m belong to the L-neighbourhood of σ_g. We denote by p and q respective projections of l and lw^m on σ_g. By replacing if necessary w by w^{-1} one can assume that 1 p, q and g are ordered in this way on σ_g.

Claim 1: $|g - q| \leqslant r$. Assume on the contrary that this assertion is false. Since g belongs to Z there is $g' \in F_{L,m} \cap \sigma_g$ such that

$$r - 9\delta \leqslant |g - g'| \leqslant r < |g - q| .$$

It follows that p and q both belong to $\sigma_{g'}$. In particular l and lw^m lie in the L-neighbourhood of $\sigma_{g'}$. This contradicts the fact that g' is (L,m)-aperiodic.

Recall that g belongs to $Z \subset F$. The point p is on σ_g. Therefore there is a point $h \in F$ which lies on σ_g between 1 and p such that $|h - p| \leqslant r$ (see Figure 3). We put $u = h^{-1}l$ and $u' = w^{-m}l^{-1}g$. Hence $g = huw^m u'$.

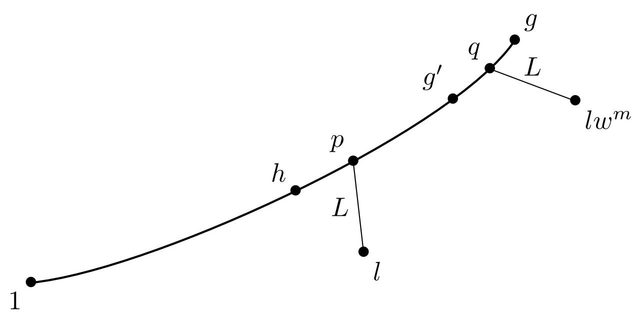

Figure 3: Positions of the points h, p, g' and q.

Claim 2: $|g| \geqslant |h| + |w^m| - 2L$. The points p and q lie on σ_g between g and h. Thus $|g| \geqslant |h| + |p - q|$. Recall that p and q are the respective projections of l and lw^m on σ_g. In particular $|l - p|, |lw^m - q| \leqslant L$. It follows from the triangle inequality that $|g| \geqslant |h| + |w^m| - 2L$.

Claim 3: The isometry h belongs to $F_{L,m}$. By construction h belongs to F. It is sufficient to prove that h is (L,m)-aperiodic. We assumed that $m\tau \geqslant 2L + r$. The previous inequality becomes

$$|g| \geqslant |h| + |w^m| - 2L \geqslant |h| + m\tau - 2L \geqslant |h| + r.$$

It follows that $|g - h| \geqslant r \geqslant |g - g'|$. In other words h belongs to $\sigma_{g'}$. Since g' is (L,m)-aperiodic, so is h.

Claim 4: $|u|, |u'| \leqslant L + r$. By construction of h,

$$|u| = |l - h| \leqslant |l - p| + |p - h| \leqslant L + r.$$

On the other hand, using our first claim,

$$|u'| = |g - lw^m| \leqslant |g - q| + |q - lw^m| \leqslant L + r.$$

Claims 2 to 4 exactly say that $g \in Z_w$, which concludes the proof. $\square$

Corollary 3.5. *For all $s \geqslant 0$,*

$$|F_{L,m} \cap B(s)| \geqslant |Z \cap B(s)| - \sum_{w \in C} |Z_w \cap B(s)|.$$

Lemma 3.6. *For all $s \geqslant 0$, $|Z \cap B(s + r)| \geqslant \rho\nu\lambda^r |F_{L,m} \cap B(s)|$.*

Remark : Recall that ρ and ν are respectively given by Lemma 2.7 and Proposition 2.14 whereas r is the radius that we fixed at the begin of this section.

Proof. Applying Lemma 2.7 there exists a subset P of $F_{L,m} \cap B(s)$ such that $|P| \geqslant \rho |F_{L,m} \cap B(s)|$ whereas for all distinct $h, h' \in P$, $|h - h'| > 20\delta$. By definition of Z,

$$\bigcup_{h \in P} h\left(L_h \cap h^{-1}F \cap A(r, 9\delta)\right) \subset Z \cap B(s + r).$$

We claim that this union is in fact a disjoint union. Assume on the contrary that this assertion is false. In particular, there are two distinct elements $h, h' \in P$ and $u, u' \in A(r, 9\delta)$ such that h and h' lie on the lexicographic geodesic from 1 to $hu = h'u'$. Since u and u' belongs to $A(r, 9\delta)$, $|h - h'| = ||h| - |h'|| \leqslant 9\delta$. However $h, h' \in P$, thus $h = h'$. Contradiction. Therefore we have

$$|Z \cap B(s + r)| \geqslant \sum_{h \in P} \left|L_h \cap h^{-1}F \cap A(r, 9\delta)\right|.$$

Nevertheless $P \subset F$. It follows from Proposition 2.14, that for all $h \in P$, $\left|L_h \cap h^{-1}F \cap A(r, 9\delta)\right| \geqslant \nu\lambda^r$. Consequently

$$|Z \cap B(s + r)| \geqslant \nu\lambda^r |P| \geqslant \rho\nu\lambda^r |F_{L,m} \cap B(s)|.$$

$\square$

Lemma 3.7. *For all $w \in G \setminus \{1\}$, for all $s \geqslant 0$,*

$$|Z_w \cap B(s+r)| \leqslant \alpha^2 \lambda^{2(L+r)} |F_{L,m} \cap B(s+r+2L-m[w]^\infty)|.$$

Proof. Let g be an element of $Z_w \cap B(s+r)$. By definition there are $h \in F_{L,m}$ and $u, u' \in B(L+r)$ such that $g = huw^m u'$. Moreover $|g| \geqslant |h| + |w^m| - 2L \geqslant |h| + m[w]^\infty - 2L$. Consequently h belongs to $B(s+r+2L-m[w]^\infty)$. Hence $Z_w \cap B(s+r)$ is a subset of

$$\Big(F_{L,m} \cap B(s+r+2L-m[w]^\infty)\Big) B(L+r)\, w^m B(L+r).$$

The conclusion follows from Proposition 1.3. □

Let us summarize. Let $s \in \mathbf{R}_+$. By Corollary 3.5 and Lemma 3.6,

$$\begin{aligned} |F_{L,m} \cap B(s+r)| &\geqslant |Z \cap B(s+r)| - \sum_{w \in C} |Z_w \cap B(s+r)| & (2)\\ &\geqslant \rho\nu\lambda^r |F_{L,m} \cap B(s)| - \sum_{w \in C} |Z_w \cap B(s+r)|. & (3)\end{aligned}$$

Let $j \in \mathbf{N}^*$. We consider $w \in C$ such that $[w]^\infty = j\tau$ (see Theorem 1.1). By Lemma 3.7,

$$|Z_w \cap B(s+r)| \leqslant \alpha^2 \lambda^{2(L+r)} |F_{L,m} \cap B(s+r+2L-mj\tau)|.$$

Since w is cyclically reduced it satisfies $|w| = [w] \leqslant [w]^\infty + 50\delta$. Hence such an isometry belongs to $B(j\tau + 50\delta)$. The number of elements $w \in C$ such that $[w]^\infty = j\tau$ is therefore bounded above by $|B(j\tau+50\delta)| \leqslant \alpha\lambda^{j\tau+50\delta}$. Consequently

$$\sum_{w \in C} |Z_w \cap B(s+r)| \leqslant \alpha^3 \lambda^{2(L+r+25\delta)} \sum_{j \geqslant 1} \lambda^{j\tau} |F_{L,m} \cap B(s+r+2L-mj\tau)|.$$

Note that in the sum on the right-hand side all but finitely many terms vanish. Combining this last inequality with (3) we get

$$\begin{aligned}&|F_{L,m} \cap B(s+r)| \geqslant \\ &\qquad \rho\nu\lambda^r |F_{L,m} \cap B(s)| - \alpha^3 \lambda^{2(L+r+25\delta)} \sum_{j \geqslant 1} \lambda^{j\tau} |F_{L,m} \cap B(s+r+2L-mj\tau)|.\end{aligned}$$

Finally we proved the following proposition.

Proposition 3.8. *There exist positive constants τ, κ_1 and κ_2 which only depends on G and A with the following property. Let $L > 0$. Let r be a real number larger than 10δ. There exists a subset F of G containing 1 such that for all integers m satisfying $m\tau > 2L + r$, for all $s \in \mathbf{R}_+$,*

$$\begin{aligned}&|F_{L,m} \cap B(s+r)| \geqslant \\ &\qquad \kappa_1 \lambda^r |F_{L,m} \cap B(s)| - \kappa_2 \lambda^{2(L+r)} \sum_{j \geqslant 1} \lambda^{j\tau} |F_{L,m} \cap B(s+r+2L-mj\tau)|.\end{aligned}$$

Before proving Proposition 3.3, we introduce a family of auxiliary maps. For all $L \geqslant 0$, for all $m \in \mathbf{N}^*$, the function $f_{L,m} : \left(\sqrt[m]{\lambda}, \lambda \right) \times \mathbf{R}_+ \to \mathbf{R}$ is given by

$$\begin{aligned} f_{L,m}(\mu, r) &= \kappa_1 \lambda^r - \kappa_2 \lambda^{4(L+r)} \sum_{j \geqslant 1} \left(\frac{\lambda}{\mu^m} \right)^{j\tau} \\ &= \kappa_1 \lambda^r - \kappa_2 \lambda^{4(L+r)} \frac{\lambda^\tau}{\mu^{m\tau} - \lambda^\tau} \end{aligned}$$

Proposition 3.9. *Let $L \geqslant 0$. There exists $m_0 \in \mathbf{N}$ such that for all integers $m \geqslant m_0$,*

$$f_{L,m} \left(\lambda \left(1 - \frac{a}{m} \right), mb \right) \geqslant \left[\lambda \left(1 - \frac{a}{m} \right) \right]^{mb}.$$

where $a = 5(1 - \kappa_1)/\tau$ and $b = \tau/4$.

Proof. For every $m \in \mathbf{N}^*$ we put $\mu_m = \lambda \left(1 - a/m \right)$. Note that if m is sufficiently large $\mu_m \in \left(\sqrt[m]{\lambda}, \lambda \right)$. Moreover we have the following properties.

$$\begin{aligned} f_{L,m} \left(\lambda \left(1 - \frac{a}{m} \right), mb \right) &= \lambda^{mb} \left[\kappa_1 + \underset{m \to +\infty}{\mathrm{o}}(1) \right], \\ (\mu_m)^{mb} &= \lambda^{mb} \left[1 - ab + \underset{m \to +\infty}{\mathrm{o}}(1) \right] \end{aligned}$$

Since $1 - ab < \kappa_1$ there exists m_0 such that for every $m \geqslant m_0$

$$f_{L,m} \left(\lambda \left(1 - \frac{a}{m} \right), mb \right) \geqslant (\mu_m)^{mb}.$$

Note that m_0 only depends G, A and L. □

Proof of Proposition 3.3. Let $L \geqslant 0$. We put $a = 5(1 - \kappa_1)/\tau$ and $b = \tau/4$. According to Proposition 3.9, $m_0 \in \mathbf{N}$ such that for all $m \geqslant m_0$.

(i) $mb \geqslant 10\delta$,

(ii) $m\tau \geqslant 2L + mb$,

(iii) $m \ln \left(\lambda \left(1 - \frac{a}{m} \right) \right) > \ln \lambda$,

(iv) $f_{L,m} \left(\lambda \left(1 - \frac{a}{m} \right), mb \right) \geqslant \left[\lambda \left(1 - \frac{a}{m} \right) \right]^{mb}$.

Let $m \geqslant m_0$. For simplicity of notation we write $\mu = \lambda \left(1 - a/m \right)$ and $r = mb$. Hence the previous inequalities can be written $r \geqslant 10\delta$, $m\tau \geqslant 2L + r$, $\lambda \mu^{-m} < 1$ and $f_{L,m}(\mu, r) \geqslant \mu^r$. By Proposition 3.8, there exists a subset F of G containing 1 such that for all $s \geqslant 0$,

$$\begin{aligned} &\left| F_{L,m} \cap B\left(s + r \right) \right| \\ &\quad \geqslant \kappa_1 \lambda^r \left| F_{L,m} \cap B\left(s \right) \right| - \kappa_2 \lambda^{2(L+r)} \sum_{j \geqslant 1} \lambda^{j\tau} \left| F_{L,m} \cap B\left(s + r + 2L - mj\tau \right) \right|. \end{aligned}$$

We now prove by induction that for all $i \in \mathbf{N}$,

$$\left| F_{L,m} \cap B\left(ir \right) \right| \geqslant \mu^r \left| F_{L,m} \cap B\left((i-1)r \right) \right|. \qquad (\mathcal{H}_i)$$

$(\mathcal{H}_0)$ is obviously true. Assume the the induction hypotheses holds for every integer smaller or equal to i. In particular, for all $t \geqslant 0$ we have

$$|F_{L,m} \cap B(ir-t)| \leqslant \mu^{-\lfloor \frac{t}{r} \rfloor r} |F_{L,m} \cap B(ir)| \leqslant \mu^{r-t} |F_{L,m} \cap B(ir)| .$$

By construction of m_0, for all $j \geqslant 1$, we have $mj\tau - 2L - r \geqslant 0$, thus

$$\begin{aligned} \lambda^{j\tau} |F_{L,m} \cap B(ir+r+2L-mj\tau)| &\leqslant \left(\frac{\lambda}{\mu^m}\right)^{j\tau} \mu^{2(r+L)} |F_{L,m} \cap B(ir)| \\ &\leqslant \left(\frac{\lambda}{\mu^m}\right)^{j\tau} \lambda^{2(r+L)} |F_{L,m} \cap B(ir)| . \end{aligned}$$

Note that m_0 has been chosen in such a way that $\lambda\mu^{-m} < 1$. Hence by summing these inequalities we obtain,

$$\begin{aligned} |F_{L,m} \cap B((i+1)r)| &\geqslant \left[\kappa_1\lambda^r - \kappa_2\lambda^{4(L+r)} \sum_{j\geqslant 1} \left(\frac{\lambda}{\mu^m}\right)^{j\tau}\right] |F_{L,m} \cap B(ir)| \\ &\geqslant f_{L,m}(\mu,r) |F_{L,m} \cap B(ir)| . \end{aligned}$$

However, by construction, $f_{L,m}(\mu,r) \geqslant \mu^r$. Consequently $(\mathcal{H}_{i+1})$ holds. A second induction shows that for all $i \in \mathbf{N}$, $|F_{L,m} \cap B(ir)| \geqslant \mu^{ir}$. Therefore the growth rate of $F_{L,m}$ is larger than $\mu = \lambda(1 - a/m)$. □

4 Growth of periodic quotients

The infiniteness of the free Burnside groups is a consequence of the following result. We will use it to estimate the growth rate of $\mathbf{B}_k(n)$.

Theorem 4.1 (S.I. Adian [1]). *Let $k \geqslant 2$. Let A be a free generating set of $\mathbf{F}_k$. There exist integers n_0 and η such that for every odd exponent $n \geqslant n_0$ the following holds. Let w be a reduced word over $A \cup A^{-1}$. If w does not contain a subword of the form $u^{\lfloor n/2 \rfloor - \eta}$ then w represents a non-trivial element of $\mathbf{B}_k(n)$.*

Theorem 4.2. *Let $k \geqslant 2$. Let A be a free generating set of $\mathbf{F}_k$. There exist $\kappa > 0$ and an integer n_0 such that for every odd exponent $n \geqslant n_0$ the exponential growth rate of $\mathbf{B}_k(n)$ with respect to the image of A is larger than*

$$(2k-1)\left(1 - \frac{\kappa}{(2k-1)^{n/2}}\right)$$

Proof. The constants n_0 and η are the ones given by Theorem 4.1. We fix $a > 2k$. According to Proposition 3.2, there exists an integer m_0 such that for every $m \geqslant m_0$ the exponential growth rate of the set of m-aperiodic words is larger than

$$(2k-1)\left(1 - \frac{a}{(2k-1)^m}\right).$$

Let $n \geqslant \max\{n_0, 2m_0 + 2\eta + 2\}$ be an odd integer. By Theorem 4.1, the natural map $\mathbf{F}_k \to \mathbf{B}_k(n)$ restricted to the set of $(\lfloor n/2 \rfloor - \eta)$-aperiodic elements is one-to-one. Therefore the growth rate of $\mathbf{B}_k(n)$ is larger than the one of this set. In particular it is larger than

$$(2k-1)\left(1 - \frac{a(2k-1)^{\eta+1}}{(2k-1)^{n/2}}\right),$$

which ends the proof. □

In [19], A.Y. Ol'shanskiĭ solved the Burnside problem for hyperbolic groups.

Theorem 4.3 (Ol'shanskiĭ [19])**.** *Let G be a non-elementary, torsion-free hyperbolic group. There exists an integer $n(G)$ such that for all odd exponents $n \geqslant n(G)$ the quotient G/G^n is infinite.*

The proof relies on the following fact. If n is large enough, then the restriction of the canonical projection $G \twoheadrightarrow G/G^n$ to a set of aperiodic elements (which is infinite) is injective. More precisely he showed the following statement.

Theorem 4.4 (Ol'shanskiĭ [19])**.** *Let G be a non-elementary, torsion-free hyperbolic group. There exist constants L, ε and $n(G)$ with the following property. Let $n \geqslant n(G)$ be an odd integer and $m \leqslant \varepsilon n$. Then the restriction of $G \twoheadrightarrow G/G^n$ to the set of (L, m)-aperiodic elements is one-to-one.*

Remarks : The definition of aperiodic words used by A.Y. Ol'shanskiĭ is slightly different from ours (see Section 3). He says that an element $g \in G$ contains a (L, m)-power if there is $(p, w) \in G \times C$ such that both p and pw^n belong to the L-neighbourhood of some geodesic between 1 and g (not necessary σ_g). However in a hyperbolic space, two geodesics joining the same extremities are 4δ-close one from the other. Hence a (L, m)-aperiodic element in the sense of Ol'shanskiĭ is $(L + 4\delta, m)$-aperiodic in our sense and conversely. Therefore the statement of Theorem 4.4 with one or the other definition are equivalent. An other approach based on the work of T. Delzant and M. Gromov [10] can be found in [8].

Theorem 4.5. *Let G be a non-elementary, torsion-free hyperbolic group and λ its exponential growth rate with respect to a finite generating set A of G. There exists a positive number κ such that for sufficiently large odd exponents n the exponential growth rate of G/G^n with respect to A is larger than $\lambda\,(1 - \kappa/n)$.*

Proof. The parameters L, ε and $n(G)$ are given by Theorem 4.4. The constants a and m_0 (which only depend on G and L) are then given by Proposition 3.3. Let $n \geqslant \left\{\varepsilon^{-1}(m_0 + 1), n(G), 2\varepsilon^{-1}\right\}$ be an odd integer. We put $m = \lfloor \varepsilon n \rfloor$. According to Proposition 3.3, there exists a subset F of G such that the exponential growth rate of $F_{L,m}$ is larger than $\lambda\,(1 - a/m) \geqslant \lambda\,(1 - 2a/\varepsilon n)$. By Theorem 4.4 the restriction of $G \twoheadrightarrow G/G^n$ to $F_{L,m}$ is one-to-one. On the other hand for every $g \in G$ the length of g with respect to A is larger than the length of its image in G/G^n with respect to the image of A. Therefore for all $r \geqslant 0$, the ball of radius r in G/G^n contains at least $|F_{L,m} \cap B\,(r)|$ elements. Thus the exponential growth rate of G/G^n is larger than the one of $F_{L,m}$. In particular it is larger than $\lambda\,(1 - 2a/\varepsilon n)$. □

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Rémi Coulon
Vanderbilt University
Department of Mathematics
remi.coulon@vanderbilt.edu